• **Finite invariance groups.** All the transformation groups G in Examples 6.1 - 6.12 are infinite, in fact continuous, groups (continuous topological groups). [Except for Example 6.1 with dimension $p = 1$, where the orthogonal group $\mathcal{O}_1 = \{\pm 1\}$.] In the next example, G is finite.

Example 6.16. (In Lehmann TSH Ch. 6, Example 7.) The data consists of independent 0-1 observations $X_1, \ldots, X_n$, where $X_i \sim \text{Bernoulli}(p_i)$. Consider the problem of testing

$$H_0 : p_1 = \cdots = p_n = \tfrac{1}{2} \quad \text{vs.} \quad H : p_1 \geq \tfrac{1}{2}, \ldots, p_n \geq \tfrac{1}{2}. \tag{6.56}$$

This problem is invariant [verify] under the permutation group $G \equiv \{\pi\}$:

$$\begin{aligned} (X_1, \ldots, X_n) &\mapsto (X_{\pi(1)}, \ldots, X_{\pi(n)}), \\ (p_1, \ldots, p_n) &\mapsto (p_{\pi_1}, \ldots, p_{\pi_n}). \end{aligned} \tag{6.57}$$

The MIS and MIP are represented by

$$\begin{aligned} T \equiv T(X_1, \ldots, X_n) &= \sum\nolimits_{i=1}^{n} X_i, \\ \tau \equiv \tau(p_1, \ldots, p_n) &= (p_{(1)} \leq \cdots \leq p_{(n)}), \end{aligned}$$

respectively, [verify!]. Clearly the distribution of T depends on $p_1, \ldots, p_n$ only through τ. The invariance-reduced problem becomes that of testing

$$H_0 : p_{(1)} = \cdots = p_{(n)} = \tfrac{1}{2} \quad \text{vs.} \quad H : \tfrac{1}{2} \leq p_{(1)} \leq \cdots \leq p_{(n)} \tag{6.58}$$

based on T. Under the null hypothesis H_0, $T \sim \text{Binomial}(n, \frac{1}{2})$ with pdf

$$f_0(k) = \frac{\binom{n}{k}}{2^n}, \qquad k = 0, \ldots, n. \tag{6.59}$$

Under an alternative $\tau = (p_1 \leq \cdots \leq p_n)$ with $\frac{1}{2} \leq p_1$, T has pdf

$$\begin{aligned} f_\tau(k) &= \sum_{\kappa} \prod_{i \in \kappa} p_i \cdot \prod_{j \notin \kappa} q_j \\ &= \prod\nolimits_{i=1}^{n} q_i \cdot \sum_{\kappa} \prod_{i \in \kappa} a_i, \end{aligned}$$

where the summations extends over all subsets $\kappa \subseteq \{1, \ldots, n\}$ of size k, $q_j = 1 - p_j$, and $a_i = \frac{p_i}{q_i}$. (Note that each $a_i \geq 1$.) Thus the MPI test for H_0 vs. the alternative τ rejects H_0 for large values of

$$\frac{f_\tau(k)}{f_0(k)} = \frac{\sum_\kappa \prod_{i \in \kappa} a_i}{\binom{n}{k}} \equiv \frac{S_k^n(a)}{\binom{n}{k}}. \tag{6.60}$$ [verify]

The numerator $S_k^n(a)$ is the *k-th elementary symmetric polynomial* in $a \equiv (a_1, \ldots, a_n)$. However, by Lemma 6.3 below, (6.60) is strictly increasing in k, so the test that rejects for large values of T is MP for H_0 vs. τ among all tests based on T. Since this test does not depend on τ, it is UMPI for the original problem (6.56). □

Lemma 6.3. If $a_1 \geq 1, \ldots, a_n \geq 1$ with at least one inequality strict, then

$$\frac{S_k^n(a)}{\binom{n}{k}} > \frac{S_{k-1}^n(a)}{\binom{n}{k-1}}. \tag{6.61}$$

Proof. Clearly (6.61) holds when $n = k$ [verify]. Thus

$$\begin{aligned}
\frac{S_k^n(a)}{\binom{n}{k}} &= \frac{\sum_\kappa S_k^k(a_\kappa)}{\binom{n}{k}\binom{k}{k}} \\
&> \frac{\sum_\kappa S_{k-1}^k(a_\kappa)}{\binom{n}{k}\binom{k}{k-1}} \\
&= \frac{(n-k+1) S_{k-1}^n(a)}{\binom{n}{k}\binom{k}{k-1}} \qquad \text{[why?]} \\
&= \frac{S_{k-1}^n(a)}{\binom{n}{k-1}}.
\end{aligned}$$ □

In (6.56) and (6.58) the null hypothesis H_0 is simple while the alternative hypothesis H is an n-dimensional *one-sided alternative.* (A similar situation occurs in Examples 6.22 and 6.36 – cf. (6.14) and (6.26)). Here, however, although H is multi-dimensional even after reduction by invariance, a UMPI test does exist. This is because $X_1, \ldots, X_n$ are discrete in fact binary, so the order statistic $U := (X_{(1)} \le \cdots \le X_{(n)})$ (the MIS under the permutation group) is equivalent to the *single statistic* $T \equiv \sum X_i$. □

Remark 6.13. It is easy to see that no UMP or UMPU test exists for the original problem (6.56). Since $X_1, \ldots, X_n$ are independent and X_i has MLR in p_i, the test based on X_i alone is MP for testing H_0 against alternatives of the form $(\frac{1}{2}, \ldots, \frac{1}{2}, p_i > \frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ and this test is unbiased for H.

Question: What if the alternative H in (6.56) is replaced by $\sum p_i > \frac{n}{2}$? □

Example 6.17. By contrast, if the data is continuous no UMPI test based on the order statistics will exist for such a one-sided alternative. For example, suppose now that $X_1, \ldots, X_n$ are independent, with $X_i \sim N(\mu_i, 1)$. [Or any one-parameter exponential family? any MLR family?] Consider the problem of testing

$$H_0 : \mu_1 = \cdots = \mu_n = \mu_0 \quad \text{vs.} \quad H : \mu_1 \ge \mu_0, \ \ldots, \ \mu_n \ge \mu_0. \tag{6.62}$$

Without loss of generality we may take $\mu_0 = 0$; otherwise replace X_i by $X_i - \mu_0$. This problem is again invariant under the permutation group, and the MIS and MIP are represented by the ordered values

$$T := (X_{(1)} \le \cdots \le X_{(n)}),$$
$$\tau := (\mu_{(1)} \le \cdots \le \mu_{(n)}),$$

respectively. The invariance-reduced problem becomes that of testing

$$H_0 : \mu_{(1)} = \cdots = \mu_{(n)} = 0 \quad \text{vs.} \quad H : 0 \le \mu_{(1)} \le \cdots \le \mu_{(n)} \tag{6.63}$$

based on T. As in Remark 6.13, no UMP or UMPU test exists for (6.62).

Exercise 6.13. (a) Show that no UMPI test exists for (6.62).

Hint: Find the pdf of T under an alternative $\upsilon = (0, \ldots, 0, a)$, $a > 0$. Show that the MPI test for this alternative depends nontrivially on a.

(b) Show that a unique LMP test exists for (6.63) in the following sense: For any fixed alternative τ, this test (not depending on τ) maximizes the derivative at $h = 0$ of the power function at local alternatives $h\tau$, $h > 0$. Thus this test is the unique LMPI test for (6.62).

(c) Use an argument similar to Remark 6.13 to show that no unique LMP test for (6.62) exists. $\bar{\square}$

Note: A. Birnbaum (*Ann. Math. Statist* 1955) and M. L. Eaton (*Ann. Math. Statist.* 1970) provide necessary conditions for the admissibility of tests for (6.62), while Charles Stein (*Ann. Math. Statist.* 1956) provides sufficient conditions.